On Monoid Congruences of Commutative Semigroups

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Abstract

Let S be a semigroup and A a subset of S. By the separator $SepA$ of A we mean the set of all elements $x \in S$ which satisfy $xA \subseteq A$, $Ax \subseteq A$, $x(S \setminus A) \subseteq (S \setminus A)$, $(S \setminus A)x \subseteq (S \setminus A)$. In this paper we characterize the monoid congruences of commutative semigroups by the help of the notion of the separator. We show that every monoid congruence of a commutative semigroup S can be constructed by the help of subsets A of S for which $SepA \neq \emptyset$.

Let S be a semigroup and A a subset of S. By the idealizer of A we mean the set of all elements x of S which satisfy $xA \subseteq A$ and $Ax \subseteq A$. The idealizer of A will be denoted by IdA. As in [2], $IdA \cap Id(S \setminus A)$ is called the separator of A and will be denoted by $SepA$.

In this paper we characterize the monoid congruences of commutative semigroups by the help of the separator. We show that a commutative semigroup S has a non universal monoid congruence if and only if $SepA \neq \emptyset$ for some subset A of S with $\emptyset \subset A \subset S$. Moreover, every monoid congruence on a commutative semigroup S can be constructed by the help of subsets A of S for which $SepA \neq \emptyset$.

Notations. Let S be a semigroup and H a subset of S. Following [1], let

$$H \ldots a = \{(x, y) \in S \times S : \; xay \in H\}, \quad a \in S$$

and

$$P_H = \{(a, b) \in S \times S : \; H \ldots a = H \ldots b\}.$$

If $\{H_i, i \in I\}$ is a family of subsets H_i of S such that $H = \cap_{i \in I} SepH_i$, then the family $\{H_i, i \in I\}$ will be denoted by $(H; H_i, I)$. For a family $(H; H_i, I) \neq \emptyset$, we define a relation $P(H; H_i, I)$ on S as follows:

$$P(H; H_i, I) = \{(a, b) \in S \times S : \; H_i \ldots a = H_i \ldots b \quad \text{for all} \quad i \in I\}.$$

For notations and notions not defined here, we refer to [1] and [2].

Theorem 1 *Let S be a semigroup and p a congruence on S. If S_k ($k \in K$) is a family of congruence classes of S modulo p, then the separator of $\cup_{k\in K} S_k$ is either empty or the union of some congruence classes of S modulo p.*

Proof. Let S_k ($k \in K$) be a family of congruence classes of S modulo p, and let $U = \cup_{k\in K} S_k$. We may assume $SepU \neq \emptyset$ and $SepU \neq S$. Then there exist elements $a, b \in S$ such that $a \in SepU$ and $b \notin SepU$. We consider an arbitrary couple (a, b) with this property, and prove that $(a, b) \notin p$. By the assumption, at least one of the following four condition holds for b:

(1.1) $bU \not\subseteq U$,

(1.2) $Ub \not\subseteq U$,

(1.3) $b(S \setminus U) \not\subseteq (S \setminus U)$,

(1.4) $(S \setminus U)b \not\subseteq (S \setminus U)$.

In case (1,1), there exists an element $c \in U$ such that $bc \notin U$. Thus $abc \notin U$, because $a \in SepU$. Since $SepU$ is a subsemigroup of S and $c \in U$, we have $aac \in U$. As U is the union of congruence classes of S modulo p, our result implies that a and b do not belong to the same congruence class of S modulo p. The same conclusion holds in cases (1.2), (1.3) and (1.4), too. From this it follows that $SepU$ is the union of congruence classes of S modulo p. □

Theorem 2 *Let S be a semigroup and H a subsemigroup of S. If $(H; H_i, I)$ is a non empty family of subsets of S, then $P(H; H_i, I)$ is a congruence on S such that the subsets H_i ($i \in I$) and H are unions of some congruence classes of S modulo $P(H; H_i, I)$.*

Proof. It can be easily verified that $P(H; H_i, I)$ is a congruence on S. Let $i \in I$ be abitrary. Assume $H_i \neq S$. Let $x, y \in S$ such that $x \in H_i$, $y \notin H_i$. Let $h \in H$. Since $H \subseteq SepH_i$, we have $hxh \in H_i$ and $hyh \notin H_i$. Thus $(x, y) \notin P(H; H_i, I)$ and so H_i is the union of some congruence classes of S modulo $P(H; H_i, I)$.

To show that H is the union of some congruence classes of S modulo $P(H; H_i, I)$ let $h \in H$ and $g \in (S \setminus H)$ be arbitrary elements. Then there is an index j in I such that $g \notin SepH_j$. From this it follows that at least one of the following holds for g:

(1.5) $gH_j \not\subseteq H_j$,

(1.6) $H_jg \not\subseteq H_j$,

(1.7) $g(S \setminus H_j) \not\subseteq (S \setminus H_j)$,

(1.8) $(S \setminus H_j)g \not\subseteq (S \setminus H_j)$.

In case (1.5), there exists an element b in H_j such that $gb \notin H_j$. Then $hgb \notin H_j$. As $hhb \in H_j$, we have $(g, h) \notin P(H; H_i, I)$. The same conclusion holds in cases (1.6), (1.7) and (1.8), too. Consequently, H is the union of some congruence classes of S modulo $P(H; H_i, I)$. Thus the theorem is proved. □

Theorem 3 *Let S be a commutative semigroup and H a subsemigroup of S. Assume that $(H; H_i, I)$ is a non empty family of subsets of S. Then $P(H; H_i, I)$ is a monoid congruence on S such that H is the identity element of $S/P(H; H_i, I)$. Conversely, every monoid congruence on a commutative semigroup can be so constructed.*

Proof. Let S be a ommutative semigroup and H a subsemigroup of S. Assume that $(H; H_i, I)$ is not empty. Then, by Theorem 2, H is a union of some congruence classes of S modulo $P(H; H_i, I)$. Let $a, b \in H$. We show that $(a, b) \in P(H; H_i, I)$. Let $i \in I$ and $x, y \in S$ be arbitrary. Assume $xay \in H_i$. Then $yxa \in H_i$ and so $yx \in H_i$, because S is commutative and $a \in H \subseteq SepH_i$. Thus $yxb \in H_i$ and so $xby \in H_i$, because $b \in H \subseteq SepH_i$. We can prove similarly that $xay \notin H_i$ implies $xby \notin H_i$. Thus $(a, b) \in P(H; H_i, I)$, indeed. Consequently, H is a congruence class of S modulo $P(H; H_i, I)$.

Next we show that H is the identity element of the factor semigroup $S/P(H; H_i, I)$. Let S_k be an arbitrary congruence class of S modulo $P(H; H_i, I)$. Let $u \in S_k$ be arbitrary. We show that, for any $a \in H$, the product ua belongs to S_k. Let $i \in I$ and $x, y \in S$ be arbitrary elements. Since S is commutative and $a \in H \subseteq SepH_i$, the product xuy belongs to H_i if and only if $xuay = xuya$ belongs to H_i. Thus $(u, ua) \in P(H; H_i, I)$ and so $ua \in S_k$. Thus H is the identity element of the factor semigroup $S/P(H; H_i, I)$, indeed.

Conversely, let S be a commutative semigroup and p a monoid congruence on S. Denote H the identity element of the factor semigroup S/p. Let $M = \cap_{k \in K} SepS_k$, where $\{S_k, k \in K\}$ is the set of all congruence classes of S

modulo p. It is clear that $H \subseteq M$. We show that $H = M$. Assume, in an indirect way, that $H \subset M$. Let $a \in H$ and $b \in M \setminus H$ be arbitrary elements. Then there is an element $k_0 \in K$ such that $b \in S_{k_0}$. As $b \in M \subseteq SepS_{k_0}$, we have $SepS_{k_0} \cap S_{k_0} \neq \emptyset$ and so $SepS_{k_0} \subseteq S_{k_0}$ (see Theorem 3 of [2]). From this it follows that $H \subseteq M \subset SepS_{k_0} \subseteq S_{k_0}$ and so $H = S_{k_0}$, because H and S_{k_0} are congruence classes of S modulo p. As $b \in S_{k_0}$, we get $b \in H$ which is a contradiction. Hence $H = M$. Consequently the congruence $P(H; S_k, K)$ is defined.

We show that $P(H; S_k, K) = p$. To show $P(H; S_k, K) \subseteq p$, let $a, b \in S$ be arbitrary elements with $(a, b) \in P(H; S_k, K)$. Let $m, n \in K$ such that $a \in S_m$, $b \in S_n$. Since H is the identity element of the factor semigroup S/p, $hah \in S_m$ and $hbh \in S_n$ for an arbitrary $h \in H$. If $n \neq m$ then $(h, h) \in S_m...a$ and $(h, h) \notin S_m...b$, because $hbh \notin S_m$. In this case $(a, b) \notin P(H; S_k, K)$ which is a contradiction. Thus $n = m$ and so $a, b \in S_m = S_n$. Consequently $(a, b) \in p$. Hence $P(H; S_k, K) \subseteq p$. As $(a, b) \in p$ implies $(xay, xby) \in p$ for all $x, y \in S$, we get $S_k...a = S_k...b$ for all $k \in K$ which implies that $(a, b) \in P(H; S_k, K)$. Consequently $p \subseteq P(H; S_k, K)$. Therefore $p = P(H; S_k, K)$. $\square$

A subset U of a semigroup S is called an unitary subset of S if, for every $a, b \in S$, the assumption $ab, b \in U$ implies $b \in U$, and also $ab, a \in U$ implies $b \in U$.

Theorem 4 *Let S be a commutative semigroup and H a subsemigroup of S. If p is a monoid congruence on S such that H is the identity of S/p, then $P(H; H_i, I) \subseteq p \subseteq P_H$, where $\{H_i, i \in I\}$ denotes the family of all subsets H_i of S satisfying $H \subseteq SepH_i$ $(i \in I)$.*

Proof. Let p be a monoid congruence on a commutative semigroup S, and let $H \subseteq S$ be the identity element of S/p. Then H is an unitary subsemigroup of S and so $H = SepH$ (see Theorem 8 of [2]). From this it follows that $H = \cap_{i \in I} SepH_i$, where $\{H_i, i \in I\}$ is the family of all subsets H_i of S for which $H \subseteq SepH_i$. Thus the congruence $P(H; H_i, I)$ is defined on S. Let $\{S_k, k \in K\}$ be the family of all congruence classes of S modulo p. By Theorem 3, $p = P(H; S_k, K)$. As $H \in (H; S_k, K) \subseteq (H; H_i, I)$, we have $P(H; H_i, I) \subseteq p \subseteq P_H$. $\square$

Corollary 5 *A commutative semigroup S has a non universal monoid congruence if and only if it has a subset A with $\emptyset \subset A \subset S$ such that $SepA \neq \emptyset$.*

Proof. Let p be a non universal monoid congruence on a commutative semigroup S and A the congruence class of S modulo p which is the identity element of the factor semigroup S/p. Then $\emptyset \subset A \subset S$. As $A \subseteq SepA$, we have $SepA \neq \emptyset$.

Conversely, let A be a subset of a commutative semigroup S such that $\emptyset \subset A \subset S$ and $SepA \neq \emptyset$. As $SepA \subseteq A$ or $SepA \subseteq (S \setminus A)$ by Theorem 3 of [2], we have $SepA \neq S$. By Theorem 3 of this paper, $SepA$ is the identity element of the factor semigroup S/P_A and so P_A is a non universal monoid congruence on S. □

References

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